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NONLINEAR ELASTIC WAVES IN RODS

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Longitudinal nonlinear waves in a solid rod with the dispersion caused by the finiteness of the rod's diameter taken into account, are investigated. The process of nonlinear distortion of the wave, which includes the formation of the stationary nonlinear impulses called solitons, is studied, and their decay with the real losses within the rod taken into account, is investigated. The simplest quantitative estimates are made. It is shown that similar processes are possible for longitudinal waves in plates. A possibility is mentioned of independent estimation of the third order elastic constants using the nonlinear distortions of longitudinal waves in rods and plates.

In the course of considering the elastic waves of finite amplitude in rods, it is found that the finite size of the rod diameter introduces a significant dispersion of the type associated with the possibility of existence of nonlinear, spherical waves including the stationary solitary impulses, i.e. solitons, the fact noted by the authors on earlier occasion (*). Transverse magnetic waves exhibit analogous dispersion [1], however in this case the nonlinearity is cubic in amplitude and can only be detected in particular cases [2]. The dispersion connected with the fact that the period of the crystal lattice is finite [3] will have an effect only at such high frequencies that the length of free path of the phonon does not exceed the wavelength. It should also be noted that the concentration of the wave energy in a small diameter rod makes it possible to increase significantly the nonlinear effects, while the waveguide properties of the rod make it possible to observe the accumulation of the nonlinear effects at a considerable distance without any influence of the diffraction spread.

1. Let us consider longitudinal elastic waves in a rod of finite diameter $2a$. Longitudinal waves present most interest here. For the transverse (bending and torsional) waves the nonlinear effects are much weaker in the case of an isotropic material, and begin to be noticeable only in the third order of magnitude. Moreover, the bending waves are strongly dispersed, and this prevents the nonlinear distorsion. We shall consider stresses under which deformations still remain elastic, and use the generally accepted expansion of the internal energy in terms of the invariants of the deformation tensor to

*) Ostrovskii, L. A. and Sutin, A. M., On acoustic solitons in solid rods and plates. Materials of the VIII-th All Union Acoustic Conference, M., 1973.

within the third order inclusive [4]

$$E = \mu u_{ik}^2 + \left(\frac{K}{2} - \frac{\mu}{3}\right) u_{ll}^2 + \frac{A}{3} u_{ik} u_{il} u_{kl} + B u_{ik}^2 u_{ll} + \frac{C}{3} u_{ll}^3 \tag{1.1}$$

Here u_i are the components of the displacement vector, u_{ik} are the components of the deformation tensor, μ denotes the shear modulus, K is the bulk modulus, while A, B and C are the third order Landau moduli.

We assume that the natural wavelength of the waves in question is much larger than the transverse dimension of the rod. We can therefore make the usual assumption that the radial displacement is proportional to the radial coordinate r and to the axial deformation, i.e.

$$u_r = -\sigma r \frac{\partial u_x}{\partial x} \tag{1.2}$$

where σ is the Poisson's ratio and the x-axis is directed along the rod. Integrating the energy density in the transverse section of the rod and taking into account (1.2), we obtain the following one-dimensional Lagrangian:

$$L = \int_S \left[\frac{1}{2}\rho\left(\frac{\partial u_i}{\partial t}\right)^2 - \varepsilon\right] dS = \frac{S\rho}{2}\left[\left(\frac{\partial u_x}{\partial t}\right)^2 + \sigma\nu\left(\frac{\partial^2 u_x}{\partial x\,\partial t}\right)^2\right] - \frac{SE}{2}\left(\frac{\partial u_x}{\partial x}\right)^2 - \frac{S\beta}{6}\left(\frac{\partial u_x}{\partial x}\right)^3 \tag{1.3}$$

$$\beta = 3E + 2A(1 - 2\sigma^3) + 6B(1 - 2\sigma + 2\sigma^2 - 4\sigma^3) + 2C(1 - 2\sigma)^3$$

Here ρ is density, S is the transverse section of the rod, E is the Young's modulus, ν is the polar radius of inertia (for a cylindrical rod $\nu = a/\sqrt{2}$ and β is the nonlinearity parameter. For most solids $\beta < 0$ [5].

The Lagrange equation corresponding to (1.3) has the form

$$\frac{\partial^2 u_x}{\partial x^2} - c^2 \frac{\partial^2 u_x}{\partial x^2} - \frac{\beta}{\rho}\frac{\partial^2 u_x}{\partial x^2}\frac{\partial u_x}{\partial x} - L^2 \frac{\partial^4 u_x}{\partial t^2\,\partial x^2} = 0 \tag{1.4}$$

$$(c^2 = E/\rho,\ L = \sigma\nu)$$

By virtue of the assumptions made, the two last terms in (1.4) describing the nonlinearity and dispersion effects, are small (although essential in what follows). The above equation was studied more than once for the case $\beta = 0$ (linear case) (see e.g. [6]). When $L = 0$ (absence of dispersion), the equation assumes the same form as that for a longitudinal wave in free space [5, 7], however in the present case the wave velocity and the nonlinearity parameter β are both functions of the parameters of the material.

To consider a nonlinear wave propagating in the direction x, we pass to the dimensionless coordinates (see also [7])

$$\tau = \frac{x}{L},\quad \xi = \frac{x - ct}{L},\quad v = -\frac{\beta}{2\rho c^3}\frac{\partial u_x}{\partial t} \tag{1.5}$$

Since the dispersion and the nonlinearity are both small, the dependence of u_x on τ should be slow compared with the dependence on ξ. Then, susbstituting (1.5) into (1.4), neglecting terms of the order of $\partial^2/\partial\tau^2$ and integrating with respect to ξ, we obtain

$$\frac{\partial v}{\partial \tau} + v\frac{\partial v}{\partial \xi} + \frac{\partial^3 v}{\partial \xi^3} = 0 \qquad (1.6)$$

A more rigorous method of deriving such equations can be found e.g. in [8].

It follows that the longitudinal velocity of the rod particles satisfies the Korteweg-de Vries equation. Its solutions were studied repeatedly in connection with various physical processes (see e.g. [9]). Let us discuss briefly the properties of the elastic waves in the rod within this approximation.

2. First we consider a sufficiently strong, low frequency wave in the rod. In this case nonlinearity prevails over dispersion ($v_0 l^2/12 \gg 1$, where v_0 and l are the amplitude and characteristic wavelength in terms of the variables (1.5)). Then the dispersion term can be neglected at the initial stage and the solution corresponds to a simple wave.

$$v = F(\xi - v\tau)$$

where F is an arbitrary function. Deformation of such a wave leads to a steepening of its front and hence, formally, to loss of uniqueness (inversion). If e.g. at $x = 0$ the variable v represents a harmonic oscillation of the form $v = v_0 \sin \omega t$ or an impulse in the form of a half period of this sine wave, i.e. $v = v_0 \sin \omega t$ for $0 < t < \pi/\omega$ and $v = 0$ outside this interval, then the wave becomes inverted at the distance $x_* = c/\omega v_0$. When $x > x_*$ a segment of large curvature appears within the wave. Within this segment dispersion can no longer be neglected and Eq. (1.6) must be used in full. Since there are no losses, we find that when $x > x_*$, oscillations will always appear at the wavefront and a wave of finite duration will split into several isolated impulses (soliton) of the form

$$v = A\,\mathrm{sech}^2\left(\frac{\xi - w\tau}{\Delta}\right)$$

where the excess (relative to the linear velocity c) wave velocity w and the spatial width of the soliton Δ (relative to the level of $0.8\,A$) are connected with the amplitude by the relations

$$A = 3w, \quad \Delta = \sqrt{12/A}$$

For example, for a cylindrical steel rod the length of the soliton in the dimensional variables is approximately equal to $0.3a/\sqrt{M}$, where $M = c^{-1}\partial u_x/\partial t$ is the acoustic Mach number.

The soliton reaches its minimum length at the largest possible value of the elastic stress for which the Hooke's law still holds. For steel the onset of plastcity corresponds to $M \approx 5\cdot 10^{-4}$ [6], and the soliton length is approximately seven times the diameter of the rod (i.e. the assumption that the transverse dimension of the rod is small compared with the wavelength holds practically always for the solitons).

The distortions in the elastic impulse and its decomposition into solitons are shown schematically in Fig. 1. The solid lines depict the dependence $v = v(\xi)$ for

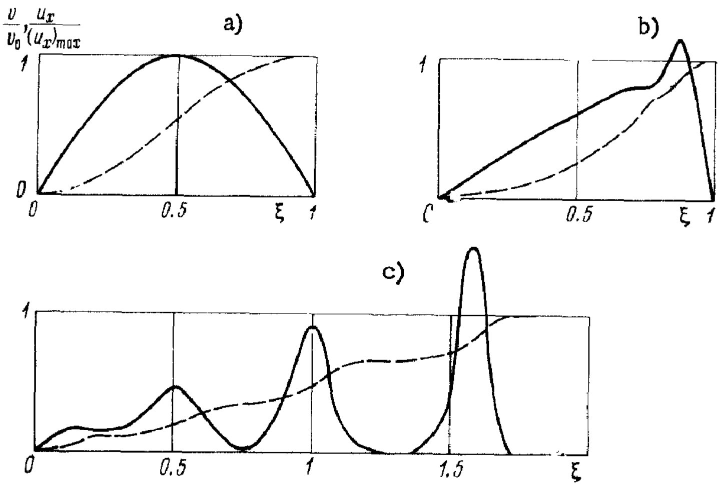

Fig. 1

$x = 0$ (a), $x = x_*$ (б) and $x > x_*$ (в). We note that the amplitude of the first soliton exceeds in magnitude the initial amplitude by about two times, and this may contribute towards the appearance of irreversible deformations within the material. A sine wave of sufficient intensity will also decompose into solitons over each period. However, at some distance it will regain its sinusoidal form and the process will be repeated periodically [10].

Until now we have always discussed the quantity v, which is proportional to the longitudinal velocity of the particles of the medium. Sometimes the longitudinal deformation represents a more characteristic quantity

$$u_x = -2\rho c^3 \beta^{-1} \int v\,dt$$

In particular, the velocity soliton has the corresponding displacement wave given in the form of a jump

$$(u_x)_{\max} = -\frac{2\rho c^3}{\beta} \int_{-\infty}^{\infty} A(\tau)\,\mathrm{sech}^2\left(\frac{\xi}{\Delta}\right) d\xi = 8\sqrt{3}\rho\sigma\nu\beta^{-1}c^3\sqrt{A(\tau)}$$

It follows that the amplitude of the velocity solitons is proportional to the square of maximum displacement, and the process of decomposition into solitons corresponds to the appearance of diverging "steps" in the displacement profile (see the dashed lines in Fig. 1). On the other hand, the transverse displacement of the lateral surface of the rod varies with time according to (1.2), in the same manner as the longitudinal velocity.

3. In order to assess the feasibility of observing the nonlinear waves, we must consider the influence of the losses within the rod. To do this, we must add to the left-hand side of (1.6) a corresponding linear operator $\mathbf{P}(v)$, the form of which depends on the mechanism of dissipation. If the linear theory supplies us with the frequency depend-

ence of the damping decrement, i. e. of the imaginary part of the wave number $k''(\omega)$ for harmonic waves, then the operator $\mathbf{P}(v)$ can be found from the inverse Fourier transformation. For example, when the losses are determined by the viscosity or heat conductivity in the rod, i. e. $k'' = \eta\omega^2$, we obtain $\mathbf{P}(v) = \eta c^2 L^{-1} \partial^2 v/\partial\xi^2$ (the value of η for longitudinal waves in a rod can be obtained from e. g. [4]). Equation (1. 6) has, in this case, the form of the Burgers-Korteweg-de Vries equation which has been recently investigated more than once. In particular, the specific features of the soliton damping when the losses are small were expounded in[11, 12]. The damping of shock waves in elastic media in the absence of dispersion was studied within the framework of the Burgers equation in [7].

Usually the damping in a solid has a different character. Experiments carried out for various media including metals [13] indicate that in the linear case the wave harmonics dampen within the rod as $\exp(-\varepsilon kx/2)$, where k is the wave number and ε is the loss coefficient which is constant. The above law holds for frequencies at which the scattering of the wave on separate crystals within the structure of the material has still no effect. For metals these frequencies extend to the value of at least several megahertz.

It can be shown that the presence of such dissipation is equivalent to the appearance in (1. 6) of a term of the type

$$\mathbf{P}(v) = \frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \frac{\partial v}{\partial \xi'} \frac{d\xi'}{\xi - \xi'} \tag{3. 1}$$

where the integral is taken in the sense of its principal value. It is interesting to note that Eq. (1. 6) with the term (3. 1) also describes the propagation of nonlinear ion-acoustic waves in a plasma with Landau damping [14, 15].

Let us present some results which are essential for the elastic waves in question. Smooth perturbations for which the dispersion term in (1. 6) can be neglected, become distorted with the steepening of the profile, to resemble a "quasisimple" wave. The dissipation however reduces the distortion and no wave inversion takes place when the amplitude is small [16]. For a sine-type input perturbations the inversion begins when

$$M > |0.08\rho c^2 \varepsilon \beta^{-1}|$$

Thus the wave amplitude must exceed a certain (frequency independent) threshold value. For example, for steel $(\varepsilon \approx 2\cdot 10^{-4})$ this value corresponds to $M > 3\cdot 10^{-7}$. At large amplitudes the dispersion begins to take effect at the steep part of the wave. Oscillations begin to appear within the wave and, as before, it decomposes into solitons. We can easily establish the law of damping of a single soliton, assuming that it retains a quasi-stationary form. The method of obtaining a solution can be reduced in this case (see [17]) to the following: integrating (1. 6) with respect to ξ from $-\infty$ to $+\infty$ we arrive at an equation for A the integral of which has the form

$$A(\tau) = A(0)\left(1 + \frac{\gamma\varepsilon\tau}{8\pi}\sqrt{\frac{A(0)}{3}}\right)^{-2} \tag{3. 2}$$

$$\gamma = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \operatorname{sech}^2 y \frac{\partial \operatorname{sech}^2 y'}{\partial y'} \frac{dy\,dy'}{(y-y')} \approx 0.725$$

The formula (3.2) holds when the dissipation is sufficiently small (more accurately, when $\varepsilon/2\pi \ll 1$) On a sufficiently large interval the amplitude of the soliton ceases to depend on its initial amplitude $(A \sim \tau^{-2})$. As a result, the amplitudes of different solitons damping towards a similar value.

We consider, as an example, the propagation of an ultrasonic wave in a 1 mm. diam. steel wire (a problem with a definite practical interest). For a wave of 100kHz frequency and a velocity amplitude of 50 cm/sec. (power of the order of 5W), the inversion distance x_* is about 8 m. when $x \sim 20$ m several solitons appear on each wave period, the largest soliton with the amplitude of about 100 cm/sec. and length of about 1 cm. Further, at a distance of about 55 m. the waveform reverts to the sinusoidal one. The process can be easily observed since the damping of such waves through dissipation begins to be apparent only at the distances of order of 100 m. (radiation dissipation into air is also small).

Similar effects can also be realized by using finite size rods (resonators), with the wave traversing it many times. In order to accumulate the nonlinear effects, the resonator must have rigid reflecting boundaries, or be shaped into a ring. We note that significant nonlinear effects (parametric generation and spectrum transformation) have already been observed experimentally in ring resonators [18].

Analogous effects are possible in thin plates where the longitudinal waves undergo the same type of dispersion as those in rods. For plates we can use Eq. (1.4) with the coefficients

$$c^2 = \frac{E}{\rho(1-\sigma^2)}, \quad L^2 = \frac{b^2\sigma^2}{12(1-\sigma)^2}$$
$$\beta = \frac{3E}{\rho(1-\sigma^2)} + A\left[1 - \left(\frac{\sigma}{1-\sigma}\right)^3\right] + 3B\left[1 - \frac{\sigma}{1-\sigma} + \left(\frac{\sigma}{1-\sigma}\right)^2 + 2\left(\frac{\sigma}{1-\sigma}\right)^3\right] + C\left(\frac{1-2\sigma}{1-\sigma}\right)^3.$$

where b denotes the plate thickness. It must be noted that even the simplest nonlinear effects (e.g. generation of the second harmonic) for longitudinal waves in extended space, rods and plates are determined, respectively, by three different combinations of three third order constants characterizing the solid. This apparently offers the possibility of measuring each of these constants separately with the help of nonlinear methods which yielded, in the past, very accurately but only one of these combinations [5], the particular combination corresponding to a plane wave in free space.

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